Engineering 227: Advanced Elasticity

Problem Set 5

Due Wednesday, November 19, 2003

1. For an incompressible material, $\sigma = -p\mathbf{I} + 2\left(\bar{W}_1 + I_1\bar{W}_2\right)\mathbf{G} - 2\bar{W}_2\mathbf{G}^2$.

a. Confirm that
the function

$$\overset{*}{w}(\lambda_1,\lambda_2,\lambda_3) = \bar{W}(I_1,I_2),\ \ I_1 = \lambda_1^2+\lambda_2^2+\lambda_3^2,\ I_2 = \lambda_1^2\lambda_2^2+\lambda_3^2\lambda_2^2+\lambda_3^2\lambda_1^2$$

allows the principal Cauchy stress components to be written as $\sigma_i = -p + \lambda_i \dfrac{\partial \overset{*}{w}}{\partial \lambda_i}(\lambda_1,\lambda_2,\lambda_3)$. (no sum on *i.*)

b. Since the function $\overset{*}{w}$ is evaluated using stretches that are constrained so that $\lambda_1\lambda_2\lambda_3 = 1$, we could just as easily have expressed that strain-energy density as a function $\overset{**}{w}$ of the principal stretches through

$$\overset{**}{w}(\lambda_1,\lambda_2,\lambda_3) = \bar{W}(I_1,I_2),\ \text{ with}$$

$$I_1 = \left(\lambda_1\lambda_2\lambda_3\right)^{-2/3}\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right) = \left(\frac{\lambda_1^2}{\lambda_2\lambda_3}\right)^{2/3} + \left(\frac{\lambda_2^2}{\lambda_1\lambda_3}\right)^{2/3} + \left(\frac{\lambda_3^2}{\lambda_2\lambda_1}\right)^{2/3},$$

$$I_2 = \left(\lambda_1\lambda_2\lambda_3\right)^{-4/3}\left(\lambda_1^2\lambda_2^2+\lambda_3^2\lambda_2^2+\lambda_3^2\lambda_1^2\right) = \left(\frac{\lambda_1\lambda_2}{\lambda_3^2}\right)^{2/3} + \left(\frac{\lambda_3\lambda_1}{\lambda_2^2}\right)^{2/3} + \left(\frac{\lambda_3\lambda_2}{\lambda_1^2}\right)^{2/3}.$$

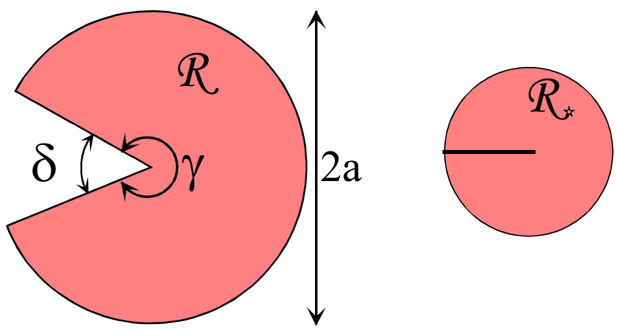
note that $\overset{**}{w}(\lambda_1,\lambda_2,\lambda_3) = \overset{*}{w}(\lambda_1/J^{1/3},\lambda_2/J^{1/3},\lambda_3/J^{1/3}),\ \ J = \lambda_1\lambda_2\lambda_3.$

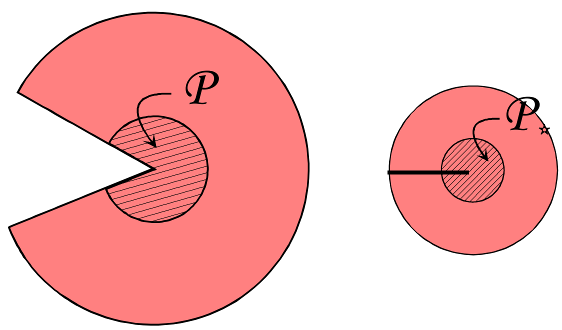
Show that the principal stress components can be written as $\sigma_i = -\overset{*}{p} + \lambda_i \dfrac{\partial \overset{**}{w}}{\partial \lambda_i}(\lambda_1,\lambda_2,\lambda_3)$ where $\overset{*}{p}$ is a kinematically indeterminate pressure term. Relate $\overset{*}{p}$ and *p,* and show that $\overset{*}{p}$ is the true pressure field.

2. The Ogden elastic potential has the form $\overset{*}{w} = \frac{2\mu}{\alpha^2}\left(\lambda_1^\alpha + \lambda_2^\alpha + \lambda_3^\alpha - 3\right)$ with μ and α constant.

The power-law material has the form $\bar{W}(I_1) = \frac{\mu}{2b}\left\{\left(1 + \frac{b}{n}(I_1 - 3)\right)^n - 1\right\}$, with μ, b, and n constant. Find the stress-stretch relation (Nominal and Cauchy) for these materials in uniaxial stress. What is Young's Modulus for each one? Plot the response for a range of the material parameters.

4. A neoHookean packman is in a state of plane strain. It is a disk of radius a with a wedge-shaped gap of angle $\delta = 2\pi - \gamma$. The gap is closed by gluing its edges together. The outer edge of the disk is traction free.





a. Assume a deformation of the form

$$x_1 = r\cos(\theta),\ x_2 = r\sin(\theta),$$
$$y_1 = R\cos(\phi),\ y_2 = R\sin(\phi),$$
$$R = \hat{R}(r),\ \phi = \hat{\phi}(\theta) = \alpha\theta.$$

Here, α is a constant. Find $\hat{r}$ and α. The incompressibility constraint determines $\hat{r}$; consider the volume of the material in the shaded region in the reference configuration (radius r) and its volume in the deformed consideration (radius $R = \hat{R}(r)$). Find α using the displacement boundary condition.

b. For this deformation, **F** is symmetric; in a cylindrical coordinate system **F** has a diagonal matrix; the on-diagonal in-plane components are λ_R and λ_ϕ. These may be obtained through geometrical considerations. For example, you can use the fact that λ_R and λ_ϕ are the stretches in the respective coordinate directions. Find these stretches in terms of the problem parameters.

c. Find the components Cauchy stress in cylindrical coordinates and express the components as function of R, ϕ. (Include the out-of-plane stress components.)

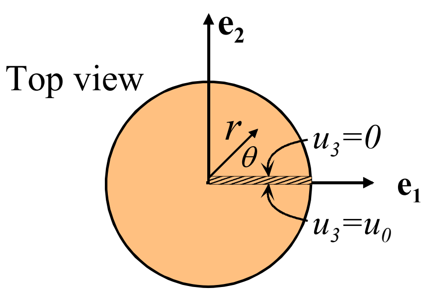
d. Assume the pressure p depends only on the radial coordinate R then the relevant in-plane Eulerian equibilrium equation in cylindrical coordinates is

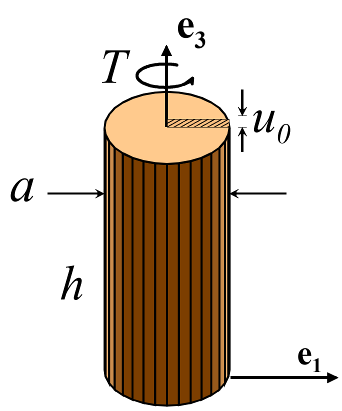
$$\frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R}\left(\frac{\partial \sigma_{R\phi}}{\partial \phi} + \sigma_{RR} - \sigma_{\phi\phi} \right) = 0.$$

Determine the Cauchy stress components.

5. A neo-Hookean rubber cylinder (radius a height h) contains a screw dislocation. The lateral boundary are traction free. The end faces have no normal tractions applied, although shear tractions provide a twisting moment of magnitude $M_3=T$ to generate the dislocation. There is no net bending moment applied to the end faces $M_1= M_2=0$. In terms of cylindrical coordinates $x_1 = r\cos(\theta),\ x_2 = r\sin(\theta),$ the slip plane is along the positive x_3 axis, and so the displacement conditions are that when $\theta = 0,\ \mathbf{u} = 0,$ and when $\theta = 2\pi,\ \mathbf{u} = u_0\mathbf{e_3}$. The tractions across the slip plane are continuous. The goal is to relate T to the slip u_0 and determine the stress fields in the material.

Assume an anti-plane shear deformation of the form $u_1 = u_2 = 0,\ u_3 = u_3(x_1, x_2) = u(r,\theta)$. There are no body forces. In what follows, p is the usual pressure field. . The elastic potential is $\overline{W}(I_1, I_2) = \frac{\mu}{2}[I_1 - 3]$.





a. Show that the Cartesian components of $\boldsymbol{P}$ in this basis are as follows:

$$\underline{P} = \mu \begin{bmatrix} 1 - p/\mu & 0 & p u_{3,1}/\mu \\ 0 & 1 - p/\mu & p u_{3,2}/\mu \\ u_{3,1} & u_{3,2} & 1 - p/\mu \end{bmatrix}.$$

b. Express the boundary conditions on $r=a$ and $\theta=0$ in terms of p, u_3 and its derivatives. Suggestion: use the cylindrical coordinates $u_3(x_1, x_2) = u(r,\theta)$.

c. You may assume that $p = \mu$. (This may be shown to be a consequence of the equilibrium conditions and boundary conditions. Use the equilibrium equations and boundary conditions to find the function u in terms of the problem parameters.

d. Confirm that the end conditions are satisfied, and find a relation between the twisting moment T and the slip u_0.

e. Determine the Cauchy stress field. Compare with the linear stress field for a screw dislocation. (look up this solution if you are not familiar with it)