222 HW3 Solutions

1) $W = \frac{\mu}{2}(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} + 2\lambda_1\lambda_2\lambda_3 - 5)$

UA stress $\quad P_{11} = \frac{\partial W}{\partial \lambda_1}(\lambda_1, \bar\lambda, \bar\lambda) = \mu(-\lambda_1^{-3} + \bar\lambda^2)$

$\sigma_{11} = \frac{1}{\bar\lambda^2} = \mu\left(-\frac{\lambda_1^{-3}}{\bar\lambda^2} + 1\right)$ true (Cauchy) stress

$P_{22} = P_{33} = 0 = \frac{\partial W}{\partial \lambda_2}(\lambda_1, \bar\lambda, \bar\lambda) = \mu(-\bar\lambda^{-3} + \lambda_1\bar\lambda) = 0$

$\Rightarrow \bar\lambda = \lambda^{-1/4}$ (Poisson's ratio $\nu = 1/4$)

$\Rightarrow \sigma_{11}(\lambda_1) = \mu(-\lambda_1^{-\frac{5}{2}} + 1)$

Young's mod. from $\quad E = W_{11}(1,1,1) - 2\nu W_{12}(1,1,1) = \frac{5}{2}\mu$

or $\quad \lambda_1 = 1 + \varepsilon_1 \quad \varepsilon_1 \ll 1$

$\Rightarrow \sigma_1(1+\varepsilon_1) = \mu(-(1+\varepsilon_1)^{-5/2} + 1)$

$\sim \underbrace{\mu\frac{5}{2}}_{E}\varepsilon_1 \quad \Rightarrow \quad E = \frac{5}{2}\mu$

a) data: $E = 4.25$ MPa $\Rightarrow$ $\mu = \frac{2}{5}E = 1.7$ Mpa.

b) Cauchy stress when $\lambda_1 = 4.2$ $\quad \sigma_1(4.2) = 1.65$ MPa

c) transverse stretch: $\bar\lambda = \lambda_1^{-1/4}$ @ break: $\bar\lambda = (4.2)^{-1/4} = .70$

area $\quad A = \bar\lambda^2 A_0 = .49\ \text{cm}^2$

d) Force applied $\quad P_{11}(4.2) = \frac{F}{A_0} = 0.81$ MPA $= .81\times10^6 \frac{N}{m^2}$

$F = .81\times10^6\ N/m^2 \times 10^{-4}\ m^2 = .81\times10^2\ N = \underline{81N}$ (18 lbs)

1. $\underset{\sim}{F} = \underset{\sim}{I} + \underset{\sim}{\beta} \qquad |\underset{\sim}{\beta}| = \varepsilon \ll 1$

a) $$P_{ij}(\underset{\sim}{I} + \underset{\sim}{\beta}) = \cancel{P_{ij}(\underset{\sim}{I})}^{\,0} + \underbrace{\frac{\partial P_{ij}}{\partial F_{k\ell}}(\underset{\sim}{I})}_{C_{ijk\ell}} \beta_{k\ell} + O(\varepsilon^2)$$

Taylor series

b) $$\underset{\sim}{\sigma} = \frac{1}{J} \underset{\sim}{P} \underset{\sim}{F}^T \qquad J = \det(\underset{\sim}{1} + \underset{\sim}{\beta}) = \underset{\sim}{I} + \operatorname{tr}\underset{\sim}{\beta} + O(\varepsilon)^2$$

$$= \frac{1}{1+\operatorname{tr}\underset{\sim}{\beta}} \underset{\sim}{P}(\underset{\sim}{I} + \underset{\sim}{\beta})^T = \underset{\sim}{P} - \underset{\sim}{P}(\operatorname{tr}\underset{\sim}{\beta}) + \underset{\sim}{P}\underset{\sim}{\beta}^T + O(\varepsilon^3)$$

since $\underset{\sim}{P}$ itself is $O(\varepsilon)$: $\underset{\sim}{\sigma} = \underset{\sim}{P} + O(\varepsilon^2)$

BAM $\Rightarrow \sigma_{ij} = \sigma_{ji} \Rightarrow C_{ijk\ell}\beta_{k\ell} = C_{jik\ell}\beta_{k\ell} \;\forall \beta \Rightarrow C_{ijk\ell} = C_{jik\ell}$

c) Hyperelastic $\quad P_{ij} = \dfrac{\partial W}{\partial F_{ij}} \qquad C_{ijk\ell} = \dfrac{\partial^2 W}{\partial F_{ij}\partial F_{k\ell}} = C_{k\ell ij}$

Note: could have $C_{ijk\ell} = C_{k\ell ij}$ even if the material is not hyperelastic. For hyperelasticity, need $\dfrac{\partial P_{ij}}{\partial F_{k\ell}}(\underset{\sim}{F}) = \dfrac{\partial P_{k\ell}}{\partial F_{ij}}(\underset{\sim}{F}) \;\forall \underset{\sim}{F}$. For major symmetry of $\underset{\sim}{C}$:

need only $\dfrac{\partial P_{ij}}{\partial F_{k\ell}}(\underset{\sim}{I}) = \dfrac{\partial P_{k\ell}}{\partial F_{ij}}(\underset{\sim}{I})$.

d) $C_{ijk\ell} = C_{jik\ell} = C_{k\ell ji} = C_{k\ell ij}$ This is the second minor symmetry

1st minor sym

major sym

major sym