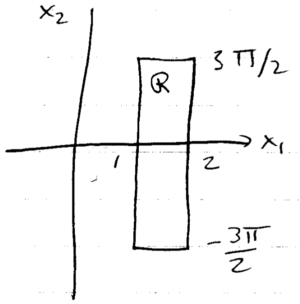
EN227 HW1

1. $\hat{y}_1(\underset{\sim}{x}) = e^{x_1}\cos x_2 \quad \hat{y}_2(x) = e^{x_1}\sin x_2 \quad \hat{y}_3(\underset{\sim}{x}) = x_3$




a) $J = \det\begin{bmatrix} \hat{y}_{1,1} & \hat{y}_{1,2} & \hat{y}_{1,3} \\ \hat{y}_{2,1} & \hat{y}_{2,2} & \hat{y}_{2,3} \\ \hat{y}_{3,1} & \hat{y}_{3,2} & \hat{y}_{3,3} \end{bmatrix} = \det\begin{bmatrix} e^{x_1}\cos & -e^{x_1}\sin & 0 \\ e^{x_1}\sin & e^{x_1}\cos & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\hat{y}_{i,j} = \frac{\partial \hat{y}_i}{\partial x_j}$

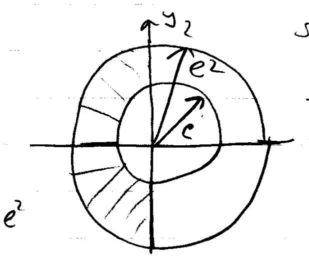
$= e^{x_1} > 0$

Locally (1-1)

b) side $x_1 = 1$, $x_2 \in \left[\frac{-3\pi}{2}, \frac{3\pi}{2}\right]$

goes to $y_1 = e\cos x_2$, $y_2 = e\sin x_2$ $\Big\rangle \sqrt{y_1^2 + y_2^2} = e$

side $x_2 = 2$ goes to $\sqrt{y_1^2+y_2^2} = e^2$




shaded region covered twice

side $x_2 = \frac{3\pi}{2}$ goes to $y_1 = 0$ $\quad y_2 \in [-e^2, -e]$

$x_2 = \frac{-3\pi}{2}$ goes to $y_1 = 0$ $\quad y_2 \in [e, e^2]$

Not globally one-to-one.

c) on the strip $R_k : \{\underset{\sim}{x} \mid (1 < x_1 < 2)\,(k < x_2 < 2\pi + k)\}$

Invert: $\sqrt{y_1^2 + y_2^2} = e^{x_1} \qquad x_1 = \frac{1}{2}\ln(y_1^2 + y_2^2)$ on $R_k$

$x_2 = \cos^{-1}\frac{y_1}{\sqrt{y_1^2+y_2^2}} = \sin^{-1}\frac{y_2}{\sqrt{y_1^2+y_2^2}} = \arg(y_1 + iy_2)$

$(k < x_2 < 2\pi + k)$

uniquely defined.

2. $(x_i - z_i)(x_i - z_i) = (\hat{y}_i(\underline{x}) - \hat{y}_i(\underline{z}))(\hat{y}_i(\underline{x}) - \hat{y}_i(\underline{z}))$

$$\frac{\partial}{\partial x_j}(\quad) = 2(x_j - z_j) = 2\frac{\partial \hat{y}_i}{\partial x_j}(\underline{x})\,(\hat{y}_i(\underline{x}) - \hat{y}_i(\underline{z}))$$

$$\frac{\partial}{\partial z_k}(\quad) = -\delta_{jk} = -\frac{\partial \hat{y}_i}{\partial x_j}(\underline{x})\,\frac{\partial \hat{y}_i(\underline{z})}{\partial z_k} = -F_{ij}(\underline{x})\,F_{ik}(\underline{z})$$

or,

$$\underline{F}^T(\underline{x})\,\underline{F}(\underline{z}) = \underline{\underline{I}} \quad \text{for all pts } \underline{x}, \underline{z} \in \mathcal{R} \quad (1)$$

Take $\underline{z} = \underline{x}$ in the above, $\underline{F}^T(\underline{x})\,\underline{F}(\underline{x}) = \underline{I} \Rightarrow \underline{F}(\underline{x}) \in \mathcal{O}_+$ (2)

now, need $F(\underline{x})$ constant.

$$(2), (1) \Rightarrow \quad \underline{F}(\underline{x}) = \underline{F}^{-T}(\underline{z}) = \underline{F}^{-T}(\underline{x}) \quad \forall\, \underline{x}, \underline{z} \quad \text{Fix } \underline{z} = \overset{\circ}{\underline{z}} \in \mathcal{R}$$

$$\underline{F}^{-T}(\underline{x}) = \underline{F}^{-T}(\overset{\circ}{\underline{z}}) \quad \forall\, \underline{x} \in \mathcal{R} \Rightarrow \underline{F}(\underline{x}) = \text{constant} = \underline{Q}$$

$$\text{so} \quad \nabla \hat{\underline{y}}(\underline{x}) = \frac{\partial \hat{\underline{y}}}{\partial \underline{x}}(\underline{x}) = \underline{\underline{Q}} \Rightarrow \hat{\underline{y}}(\underline{x}) = \underline{Q}\underline{x} + \underline{c}.$$

3. $\hat{\underline{\sigma}} = (\det \underline{F})^{-1}\,\underline{F}\underline{M}\underline{F}^T - (\det \underline{F})^{\beta}\,m(\underline{x})\,\underline{\underline{I}}$

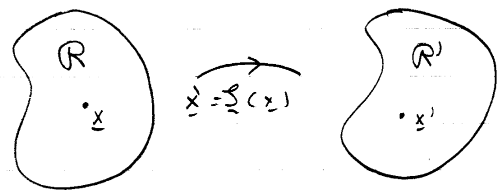
a) $\hat{\underline{\sigma}}(\underline{Q}\underline{F}) = (\det \underline{Q}\underline{F})^{-1}\,\underline{Q}\underline{F}\underline{M}\underline{F}^T\underline{Q}^T - (\det \underline{Q}\underline{F})^{\beta}\,m(\underline{x})\,\underline{\underline{I}} \qquad \underline{Q} \in \mathcal{O}_+$

$$= \underline{Q}\left[(\det \underline{F})^{-1}\,\underline{F}\underline{M}\underline{F}^T - (\det \underline{F})^{\beta}\,m(\underline{x})\,\underline{\underline{I}}\right]\underline{Q}^T \quad \checkmark$$

b) $\hat{\underline{P}}(\underline{F}) = \det \underline{F}\,\hat{\underline{\sigma}}(\underline{F})\,\underline{F}^{-T} = \underline{F}\underline{M} - (\det \underline{F})^{\beta+1}\,m\,\underline{F}^{-T}$

c) want a stress - free configuration

Change of config $\hat{\sigma}'(F) = \hat{\sigma}(FH)$ $H$ is the gradient of the map from $\mathcal{R} \to \mathcal{R}'$




$\hat{\sigma}'(I) = 0$ if $\hat{\sigma}(H) = 0$. So solve for $H$. Try $H = cB^{-1}$

$$\hat{\sigma}(cB^{-1}) = (\det(cB^{-1}))^{-1} c^2 B^{-1} B^2 B^{-T} - (\det cB^{-1})^{\beta} I = 0$$

solve for c

$\det(cB^{-1}) = c^3/\det B \Rightarrow$

$$\hat{\sigma}(cB^{-1}) = \underbrace{\left(\frac{\det B}{c^3} c^2 - c^{3\beta}(\det B)^{-\beta}_{m}\right)}_{=0} I = 0$$

$$c^{3\beta+1} = (\det B)^{\beta+1}/m \Rightarrow c$$

$$c = \left(\frac{(\det B)^{\beta+1}}{m}\right)^{\frac{1}{1+3\beta}}$$

$H = (\det B)^{\frac{\beta+1}{3\beta+1}} B^{-1}$ is a map that carries $\mathcal{R}$ to a stress free config, $\mathcal{R}'$

$$\hat{\sigma}'(F) = \hat{\sigma}(FcB^{-1}) = \frac{\frac{1}{c}\det B}{\det F} F F^{T} - (\det F)^{\beta} m c^{3\beta} (\det B)^{-\beta} I$$

$$= (\det B)^{\frac{2\beta}{3\beta+1}} \left( \frac{1}{\det F} F F^{T} - (\det F)^{\beta} I \right)$$

d) $\underset{\sim}{M} = \overbrace{\frac{3}{2}\left(\frac{b}{r}\right)^3}^{c_1(r)} \frac{x \otimes x}{r^2} - \overbrace{\left[\frac{1}{2}\left(\frac{b}{r}\right)^3 + 1 - m(x)\right]}^{c_2(r)} \underset{\sim}{I}$

MSG $\hat{\sigma}(FH) = \hat{\sigma}(F) \ \forall F \in \mathcal{L}_+$

$$\hat{\sigma}(FH) = (\det F)^{-1} \det H^{-1} F H M H^T F - (\det F)^{\beta} (\det H)^{\beta} m I$$

$$= (\det F)^{-1} \det H^{-1} F M F^T - (\det F)^{\beta} m I \quad \forall F \in \mathcal{L}_+$$

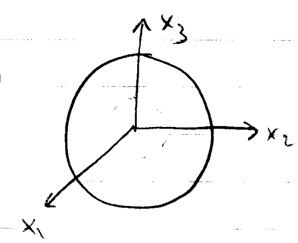
$\det H = 1$ needed, also $H M H^T = M$

works if $H \in \mathcal{O}_+$ & $Hn = n$, $n = \frac{x}{R}$ (rotation about $\frac{x}{r}$)

Then $H M H^T = c_1 H \frac{x}{r} \otimes \frac{Hx}{r} - c_2 H H^T$

$$= c_1 \frac{x}{r} \otimes \frac{x}{r} - c_2 I = M$$

Transverse orthotropy.

e) 


in the ref config $\hat{y}(x) = x = y$. $F = I$

$$\sigma(y) = \hat{\sigma}(I, x)\Big|_{x=y} = c_1(r) \frac{y \otimes y}{r^2} - (c_2(r) + m(x)) I$$

$\sigma(y) = c_1(r) \frac{y \otimes y}{r^2} - \underbrace{\left(\frac{1}{2}\left(\frac{b}{r}\right)^3 + 1\right)}_{d(r)} I \qquad r = |y| \qquad \frac{\partial r}{\partial y_i} = \frac{y_i}{r}$

$\sigma_{ij} = c_1 \frac{y_i y_j}{r^2} - d\,\delta_{ij} \qquad \sigma_{ij,j} = \frac{dc_1}{dr} \frac{y_j}{r} \frac{y_i y_j}{r^2} + c_1\left(\frac{\delta_{ij} y_j}{r^2} + \frac{y_i \delta_{jj}}{r^2}\right)$

$$- c_1 \frac{2}{r^2} \frac{y_j}{r} \frac{y_i y_j}{r^2} - d'(r) \frac{y_i}{r}$$

$$\sigma_{ij,j} = \frac{dc_1}{dr}\frac{y_i}{r} + 4\frac{c_1}{r}\frac{y_i}{r} - 2\frac{c_i y_i}{r^2} - d'(r)\frac{y_i}{r}$$

$$= \left(-\frac{9}{2}\frac{b^3}{r^4} + 3\frac{b^3}{r^4} + \frac{3}{2}\frac{b^3}{r^4}\right)\frac{y_i}{r} = 0 \quad \checkmark \quad \text{phew}$$

BC: on $r=b$ $\quad \underline{n} = \underline{y}/b \quad \underset{\sim}{\sigma}\,\underline{n} = c_1(b)\,\underline{y}/b - d(b)\,\underline{y}/b$

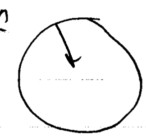
$$= (c_1(b) - d(b))\,\underline{y}/b$$

$c_1(b) - d(b) = \frac{3}{2} - \frac{3}{2} = 0 \qquad \underset{\sim}{\sigma}\,\underline{n} = \underline{0}$ on $r=b$

on $r=a \quad \underline{n} = \frac{-\underline{y}}{a} \quad \underset{\sim}{\sigma}\,\underline{n} = (-c_1(a) + d(a))\,\underline{y}/a$

$$-c_1(a) + d(a) = -\frac{3}{2}\left(\frac{b}{a}\right)^3 + \frac{1}{2}\left(\frac{b}{a}\right)^3 + 1 = 1 - \left(\frac{b}{a}\right)^3$$

$$\underset{\sim}{\sigma}\,\underline{n} = \left(1 - \left(\frac{b}{a}\right)^3\right)\frac{\underline{y}}{a} = \left(\left(\frac{b}{a}\right)^3 - 1\right)\underline{n}$$



$a < b \Rightarrow \frac{b}{a} - 1 > 0$

tension − vacuum inside.