Name: Solutions

1. (33 points) Find the solution $u(x,y)$ to the initial value problem:

$$xyu_x + u_y = y,\ \ y \ge 0,\ (-\infty < x < \infty),\ u(x,0) = u_0(x).$$

The answer will involve the function u_0. Be sure to write the ans as $u(x,y)$

$\bar{x}_\xi = \bar{x}\bar{y} \qquad \bar{y}_\xi = 1 \qquad \bar{u}_\xi = \bar{y}$

$\bar{x}(0,\eta) = x_0(\eta) = \eta \qquad \bar{y}(0,\eta) = 0 \qquad \bar{u}(0,\eta) = u_0(\eta)$

$\bar{y}(\xi,\eta) = \xi \qquad \bar{u}_\xi = \xi \Rightarrow \bar{u} = \frac{1}{2}\xi^2 + u_0(\eta)$

$\bar{x}_\xi = \bar{x}\xi \qquad (\ln \bar{x})_\xi = \xi \Rightarrow \bar{x} = \eta e^{\frac{1}{2}\xi^2}$

$x = \eta e^{\frac{1}{2}y^2} \qquad \eta = x e^{-\frac{1}{2}y^2}$

$$\boxed{u(x,y) = \frac{1}{2}y^2 + u_0(xe^{-\frac{1}{2}y^2})}$$

$u(x,0) = u_0(x)$

$u_x = u_0' e^{-\frac{1}{2}y^2}$

$u_y = y - xye^{-\frac{1}{2}y^2} u_0'$

$xyu_x + u_y = \cancel{xyu_0' e^{-\frac{1}{2}y^2}} + y - \cancel{xye^{-\frac{1}{2}y^2} u_0'} = y$ ✓

2. (33 points) A simple model for the propagation of wind-driven water waves in a channel is given below.

$$(2+u)u_x + u_y = 0, \ (-\infty < x < \infty, \ y \geq 0).$$

For the initial condition $u(x,0) = \sin(x)$, find the value of y for which the waves begin to break (a vertical tangent appears). What are the positions of the vertical tangents?

$\bar{x}_\xi = 2 + \bar{u}, \quad \bar{y}_\xi = 1, \quad \bar{u}_\xi = 0$

$\bar{x}(0,\eta) = \eta \quad \bar{y}(0,\eta) = 0 \quad \bar{u}(0,\eta) = \sin\eta$

$\bar{x} = (2 + u_0(\eta))\xi + \eta \quad y = \xi$

$\boxed{x = (2 + u_0(\eta))y + \eta = (2 + \sin\eta)y + \eta}$ characteristics

$u = \bar{u} = u_0(\eta) = \sin\eta$

Vertical tangents $u_x = \cos\eta \, \eta_x \to \infty \Rightarrow \eta_x \to \infty$

$1 = (\cos\eta \, y + 1)\eta_x$

$\eta_x \to \infty \quad y = \frac{-1}{\cos\eta}$ min when $\cos\eta = -1$ $\eta = \pi, 3\pi, 5\pi$

$\eta = 2\pi(n+1) \quad n = 0, \pm 1, \pm 2, \ldots$

$y_b = 1 \quad x_b = (2 + \cancel{\sin\eta}^{\,0})y_b + \eta$

$x_b = 2y_b + 2\pi(n+1) \quad n = 0, \pm 1, \pm 2, \ldots$

$x_b = 2 + 2\pi(n+1) \quad n = 0, \pm 1, \pm 2, \ldots$

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3. (33 points) Here is another equation:

$$u_{xx} - u_{yy} + u_x - u_y = 0, \ (y \ge 0, -\infty \le x \le \infty). \qquad (*)$$

   a. Find expressions for the characteristic coordinates ξ and η.
   b. Find an equation satisfied by the function $\bar{u}(\xi,\eta)$.
   c. Solve the equation found above. You will find that the answer involves 2 undetermined functions, $f(\xi)$ and $g(\eta)$.
   d. The solution to the PDE may be written as

$$u(x,y) = e^{-y}\phi(x-y) + \psi(x+y),$$

   where ϕ and ψ are scalar functions of their arguments.
   - If you got a solution in part c, reconcile your solution with the representation above.
   - If you did not get a solution in part c, show by substitution that u given above satisfies the PDE $(*)$.

   e. Determine the functions ϕ and ψ so that the solution satisfies the initial conditions $u(x,0) = \cancel{u_0(x)}$ e^x, $u_y(x,0) = 0$, $(-\infty \le x \le \infty)$.

a) $\frac{dy}{dx} = \frac{1}{2}\left\{\pm\sqrt{4}\right\} = \pm 1 \qquad \xi = x+y, \ \eta = x-y$

b) $\bar{u}(x+y, x-y) = u(x,y)$

$u_x = \bar{u}_\xi + \bar{u}_\eta \qquad u_y = \bar{u}_\xi - \bar{u}_\eta$

$u_{xx} = \bar{u}_{\xi\xi} + 2\bar{u}_{\xi\eta} + \bar{u}_{\eta\eta} \qquad u_{yy} = \bar{u}_{\xi\xi} - 2\bar{u}_{\xi\eta} + \bar{u}_{\eta\eta}$

$u_{xx} - u_{yy} + u_x - u_y = 4\bar{u}_{\xi\eta} + \cancel{\bar{u}_\xi} + \bar{u}_\eta - \cancel{\bar{u}_\xi} + \bar{u}_\eta$

$\boxed{\bar{u}_{\xi\eta} + \frac{1}{2}\bar{u}_\eta = 0}$ PDE

c) put $\bar{w} = \bar{u}_\eta \qquad \bar{w}_\xi + \frac{1}{2}\bar{w} = 0 \longrightarrow \ln\bar{w} = -\frac{1}{2}\xi + F(\eta)$

$\bar{w}(\xi,\eta) = m(\eta)e^{-\frac{1}{2}\xi} = \bar{u}_\eta \Rightarrow \bar{u} = \int \bar{w}(\xi,\eta)\,d\eta + h(\eta)$

$\bar{u} = e^{-\frac{1}{2}\xi}f(\eta) + g(\xi)$

$\underline{u(x,y) = e^{-\frac{1}{2}(x+y)} f(x-y) + g(x+y)}$

d. put $f(x-y) = e^{+\frac{1}{2}(x-y)}\phi(x-y), \ g(x+y) = \psi(x+y)$

$u(x,y) = e^{-\frac{1}{2}x - \frac{1}{2}y + \frac{1}{2}x - \frac{1}{2}y}\phi(x-y) + \psi(x+y)$

$\boxed{u(x,y) = e^{-y}\phi(x-y) + \psi(x+y)}$

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Check by subst

$u_x = e^{-y}\phi' + \psi'$ $\qquad$ $u_y = -e^{-y}\phi - e^{-y}\phi' + \psi'$

$u_{xx} = e^{-y}\phi'' + \psi''$ $\qquad$ $u_{yy} = e^{-y}\phi + 2e^{-y}\phi' + e^{-y}\phi'' + \psi''$

$u_{xx} - u_{yy} + u_x - u_y =$

$$\cancel{e^{-y}\phi''} + \cancel{\psi''} - \cancel{e^{-y}\phi} - \cancel{2e^{-y}\phi'} - \cancel{e^{-y}\phi''} - \cancel{\psi''}$$
$$+ \cancel{e^{-y}\phi'} + \psi' + \cancel{e^{-y}\phi} + \cancel{e^{-y}\phi'} - \psi' = 0 \checkmark$$

e. $u(x,0) = \phi(x) + \psi(x) = u_0(x) = e^x \quad (1)$

$u_y(x,0) = -\phi(x) - \phi'(x) + \psi'(x) = 0 \quad (2)$

diff (1) $\qquad \phi' + \psi' = e^x$

use (2) $\qquad 2\phi' + \phi = e^x \qquad \phi' + \frac{1}{2}\phi = \frac{1}{2}e^x$

$$\left(e^{\frac{1}{2}x}\phi\right)' = e^{\frac{1}{2}x}\left(\phi' + \tfrac{1}{2}\phi\right) = \tfrac{1}{2}e^{\frac{1}{2}x}e^x = \tfrac{1}{2}e^{3x/2}$$

$$e^{\frac{1}{2}x}\phi = \frac{1}{3}e^{3x/2} + C \leftarrow \text{set to zero}$$

$$\phi = \frac{1}{3}e^x$$

$$\psi = e^x - \phi = \frac{2}{3}e^x$$

$$u(x,y) = e^{-y}\frac{1}{3}e^{x-y} + \frac{2}{3}e^{x+y}$$

$$\boxed{u(x,y) = \frac{1}{3}e^{x-2y} + \frac{2}{3}e^{x+y}}$$

$u(x,0) = e^x \checkmark$

$u_y(x,y) = -\frac{2}{3}e^{x-y} + \frac{2}{3}e^{x+y} = 0$ for $y=0$ $\checkmark$

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