Name:________________________________

1. (35 points) For the initial value problem:

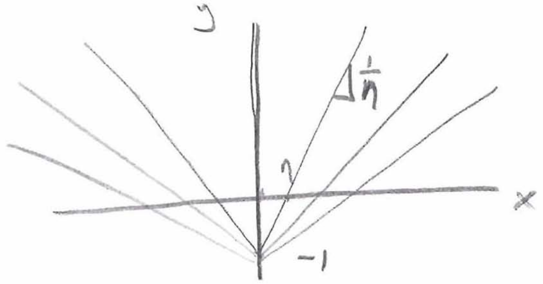
$$xu_x + (y+1)u_y = 0,\ \ y \ge 0,\ (-\infty < x < \infty),\ u(x,0) = x$$

   a. Find the characteristics for the partial differential equation.
   b. Use your characteristics to find a solution to the initial value problem.

a. $\bar{x}_s = \bar{x}$ $\quad \bar{y}_s = \bar{y}+1$ $\quad \bar{x}(0,\eta) = \eta$ $\quad \bar{y}(0,\eta) = 0$

$\Rightarrow$ $\bar{x}(s,\eta) = \eta e^s$ $\quad \bar{y}(s,\eta) = e^s - 1$

or $\boxed{y = \frac{x}{\eta} - 1 = \frac{x-\eta}{\eta}}$




b. Solve the IVP

$U_s = U_x \bar{x}_s + U_y \bar{y}_s = xU_x + (y+1)U_y = 0$ $\quad U(0,\eta) = \eta$

$U = \eta \Rightarrow u(x,y) = \eta(x,y)$ but $y = \frac{x}{\eta} - 1 \Rightarrow$

$\eta = \frac{x}{1+y} \Rightarrow \boxed{u(x,y) = \frac{x}{1+y}}$

Check $u(x,0) = x$ ✓ $\quad u_x = \frac{1}{1+y}$ $\quad u_y = \frac{-x}{(1+y)^2}$

$xu_x + (1+y)u_y = 0$ ✓

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2. (30 points) For the equation

$$u_{xx} - 4u_{xy} + 3u_{yy} = 0 \ (*)$$

a. Characteristic coordinates for this equation are of $\xi = x + c_1 y,\ \eta = x + c_2 y$ where $c_1$ and $c_2$ are constants. Determine $c_1$ and $c_2$ from the partial differential equation.

b. Use these coordinates to write a solution to the equation with initial conditions

$$u(x,0) = u_0(x),\ u_y(x,0) = 0\ (-\infty < x < \infty).$$

You may take for granted that if $U(\xi,\eta) = u(x,y)$, equation (*) reduces to $U_{\xi\eta} = 0$.

a) $a = 1 \quad b = -4 \quad c = 3$

characteristics $\quad \frac{dy}{dx} = \frac{1}{2}\left[-4 \pm \sqrt{16-12}\right] = \frac{1}{2}\left[-4 \pm 2\right] = -3, -1$

So $\quad y + 3x = \text{const}$ on characteristics & $y + x = \text{const}$ on characteristics

$c_1 = 1, \quad c_2 = 1/3$

b) $U(\xi,\eta) = u(x,y) \qquad \xi = x + y \quad \eta = x + 1/3\, y$

$u_x = U_\xi + U_\eta \qquad u_{xx} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$

$u_{xy} = U_{\xi\xi} + U_{\xi\eta} + \frac{1}{3}U_{\xi\eta} + \frac{1}{3}U_{\eta\eta} = U_{\xi\xi} + \frac{2}{3}U_{\xi\eta} + \frac{1}{3}U_{\eta\eta}$

$u_y = U_\xi + \frac{1}{3}U_\eta \qquad u_{yy} = U_{\xi\xi} + \frac{2}{3}U_{\xi\eta} + \frac{1}{9}U_{\eta\eta}$

$\Rightarrow u_{xx} - 4u_{xy} + 3u_{yy} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$

$\qquad - 4U_{\xi\xi} - \frac{8}{3}U_{\xi\eta} - \frac{4}{3}U_{\eta\eta}$

$\qquad + 3U_{\xi\xi} + 2U_{\xi\eta} + \frac{1}{3}U_{\eta\eta} = 0 \Rightarrow U_{\xi\eta} = 0$

$U = g(\xi) + h(\eta) \qquad u(x,y) = g(x+y) + h(x + \frac{1}{3}y)$

$u(x,0) = u_0(x) = g(x) + h(x) \qquad u_y(x,0) = 0 = g'(x) + \frac{1}{3}h'(x) = 0$

$\Rightarrow g(x) + \frac{1}{3}h(x) = K \quad \& \quad h(x) = \frac{3}{2}(u_0(x) - K) \quad g(x) = K - \frac{1}{3}h(x)$

$g(x) = -\frac{1}{2}u_0(x) + \frac{3}{2}K$

$$\boxed{u(x,y) = \frac{1}{2}\left[3u_0(x + \tfrac{1}{3}y) - u_0(x+y)\right]}$$

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3. (35 points) Find the solution to the initial value problem:

$$uu_x + u_y = 1, \quad y \geq 0, \ (-\infty < x < \infty), \ u(x,0) = u_0(x) = x$$

characteristics $\bar{x}_\xi = u \quad \bar{x}(0,\eta) = \eta \quad \bar{y}_\xi = 1 \quad \bar{y}(0,\eta) = 0$

& The PDE $\bar{U}_\xi = U_x \bar{x}_\xi + U_y \bar{y}_\xi = uu_x + u_y = 1 \Rightarrow \bar{U}_\xi = 1 \quad \bar{U}(0,\eta) = \eta$

$\bar{U}_\xi = 1 \Rightarrow \bar{U} = \xi + \eta$ since $\bar{U}(0,\eta) = \eta$

$\bar{x}_\xi = U = \xi + \eta \Rightarrow \bar{x}(\xi,\eta) = \frac{1}{2}\xi^2 + \xi\eta + \eta$ since $\bar{x}(0,\eta) = \eta$

$\bar{y}_\xi = 1 \Rightarrow \bar{y}(\xi,\eta) = \xi$ since $\bar{y}(0,\eta) = 0$

Solution is $u(x,y) = \xi(x,y) + \eta(x,y)$

with $y = \xi$ & $x = \frac{1}{2}\xi^2 + \xi\eta + \eta = \frac{1}{2}y^2 + \eta(y+1) \Rightarrow$

$$\eta(x,y) = \frac{x - \frac{1}{2}y^2}{y+1}$$

$$\boxed{u(x,y) = y + \frac{x - \frac{1}{2}y^2}{y+1} = \frac{\frac{1}{2}y^2 + x + y}{y+1}}$$

Check: $u(x,0) = x$ ✓ $\quad u_x = \frac{1}{1+y} \quad u_y = 1 - \frac{y}{1+y} - \frac{x - \frac{1}{2}y^2}{(1+y)^2}$

$$uu_x + u_y = \frac{y}{1+y} + \frac{x - \frac{1}{2}y^2}{(y+1)^2} + 1 - \frac{y}{1+y} - \frac{x - \frac{1}{2}y^2}{(1+y)^2} = 1 \checkmark$$