1. (25 points) Look for a similarity solution of the form $u(x,y) = y^\alpha f(\eta)$, $\eta = x/y^\beta$ to the PDE:

$$u_{xx} - xu_y = 0, \; (-\infty < x < \infty, \; y \leq \infty)$$

a. Determine conditions on α and β and obtain an ODE for f.

b. Find the solution to the PDE with the conditions
$$u(0,y) = 1 \; (y > 0) \text{ and } u(x,0) = 0 \; (x > 0)$$

a) $u = y^\alpha f(\eta) \quad \eta = x/y^\beta$

$u_x = y^{\alpha-\beta} f' \quad u_{xx} = y^{\alpha-2\beta} f'' \quad u_y = y^{\alpha-1}(\alpha f - \beta\eta f')$

PDE is $\quad y^{\alpha-2\beta} f'' - x y^{\alpha-1}(\alpha f - \beta\eta f') = 0$

$x = \eta y^\beta \Rightarrow$

$y^{\alpha-2\beta} f'' - \eta y^{\alpha-1+\beta}(\alpha f - \beta\eta f') = 0$

$\Rightarrow \quad \alpha - 2\beta = \alpha - 1 + \beta \qquad \underline{\beta = 1/3}$

ODE $\quad f'' - \eta(\alpha f - \frac{1}{3}\eta f') = 0$

b) $u(0,y) = y^\alpha f(0) = 1 \quad y > 0 \Rightarrow \alpha = 0 \;\&\; f(0) = 1$

$u(x,0) = 0 = \lim_{\eta\to\infty} f(\eta) = 0$

$\alpha = 0 \Rightarrow$ PDE is $\quad f'' + \frac{1}{3}\eta^2 f' = 0 \qquad v = f'$

$v' + \frac{1}{3}\eta^2 v = 0 \quad \Rightarrow \quad v = c_1 e^{-\frac{1}{9}\eta^3}$

$f(\eta) = c_1 \int_0^\eta e^{-\frac{1}{9}s^3} ds + 1$ since $f(0) = 1$

$f \to 0$ as $\eta \to \infty$

$c_1 \int_0^\infty e^{-\frac{1}{9}s^3} ds = -1 \qquad c_1 = \dfrac{-1}{\int_0^\infty e^{-\frac{1}{9}s^3} ds}$

Solution $\boxed{u = \dfrac{-1}{\int_0^\infty e^{-\frac{1}{9}s^3} ds} \int_0^{x/y^{1/3}} e^{-\frac{1}{9}s^3} ds + 1}$

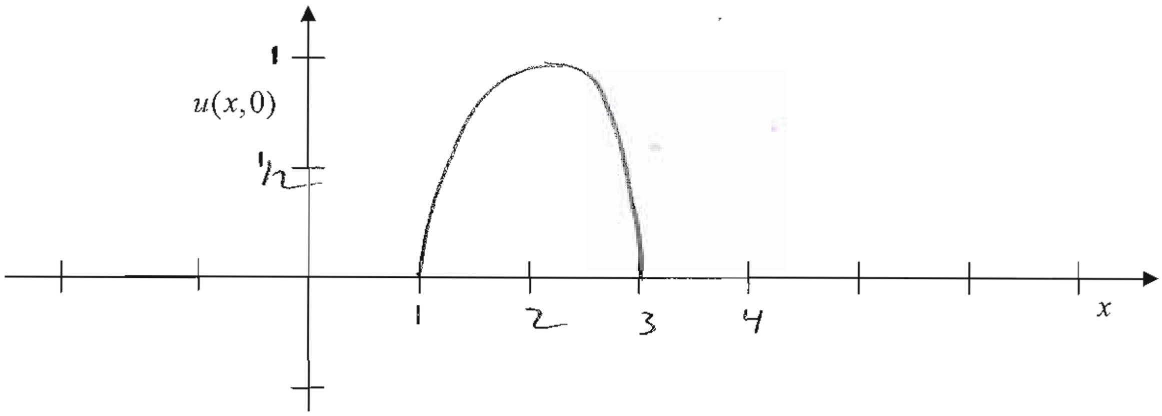
2. (10 points) Consider the wave equation on the half line with

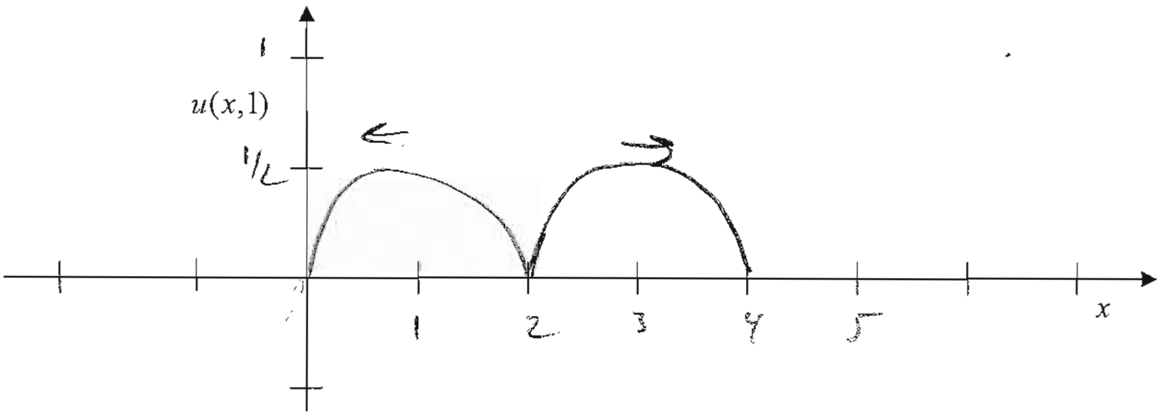
$$u_{xx} - u_{yy} = 0 \ (0 < x < \infty,\ y > 0),$$

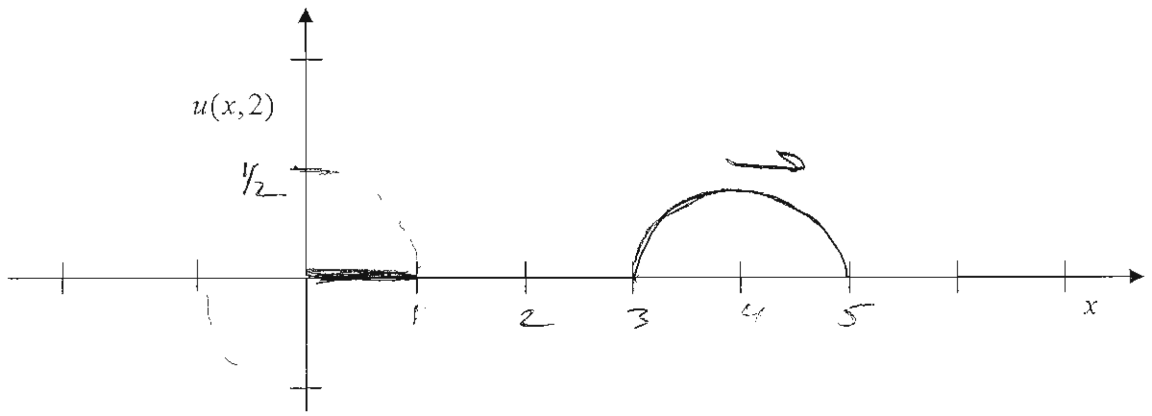
$$u(x,0) = \begin{cases} \sin\left(\frac{\pi}{2}(x-1)\right) & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases},$$

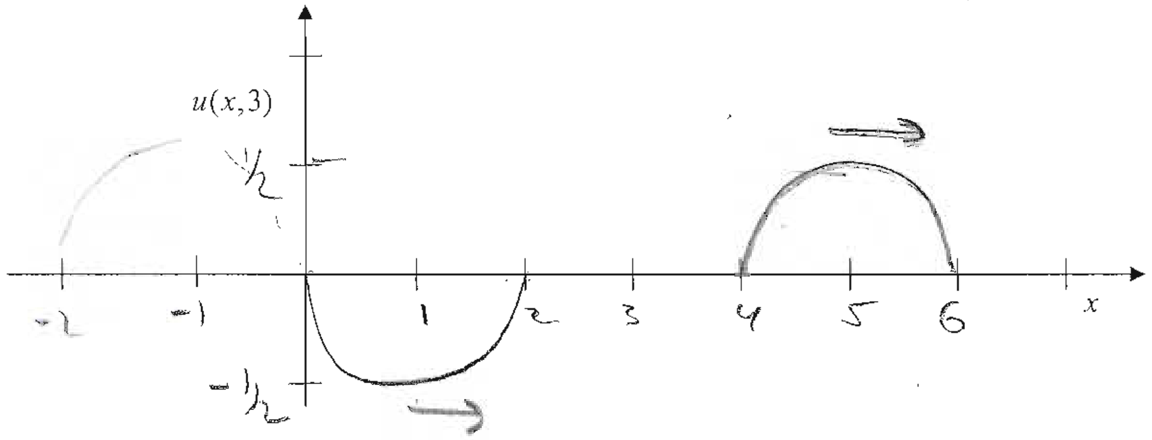
$$u_y(x,0) = 0,\ (x > 0) \text{ and } u(0,y) = 0 \qquad y > 0$$

On the graphs below and on the next page, sketch of the solution at y=0, 1, 2, and 3. Label the tick marks.






$u(x,2)$
1/2
1
2
3
4
5
x


$u(x,3)$
1/2
-1/2
-2
-1
1
2
3
4
5
6
x

3. (25 points) Use the Fourier Transform to find the solution to the following IVP:

$$u_{xx} - u_y = \delta(x-y) \quad (-\infty < x < \infty,\ y > 0)$$

$$u(x,0) = 0, \quad (-\infty < x < \infty)$$

You may leave the answer in the form of a single integral.

transform the PDE $\hat{u} = \int_{-\infty}^{\infty} e^{+i\omega x} u(x,y)\,dx$

$$\underbrace{\int_{-\infty}^{\infty} e^{+i\omega x} u_{xx}(x,y)\,dx}_{} - \int_{-\infty}^{\infty} e^{+i\omega x} u_y(x,y)\,dx = \int_{-\infty}^{\infty} \delta(x-y) e^{+i\omega x}\,dx$$

$$\cancel{e^{+i\omega x} u_x \Big|_{-\infty}^{\infty}} - \cancel{i\omega u \Big|_{-\infty}^{\infty}} - \omega^2 \int_{-\infty}^{\infty} e^{-i\omega x} u\,dx - \hat{u}_y = e^{+i\omega y}$$

$$\hat{u}_y + \omega^2 \hat{u} = -e^{-i\omega y} \rightarrow \hat{u} = \underset{\substack{\uparrow \\ \text{homog}}}{c e^{-\omega^2 y}} + \underset{\substack{\uparrow \\ \text{part}}}{u_p}$$

particular solution $u = A e^{+i\omega y}$

$$A(+i\omega + \omega^2) e^{+i\omega y} = -1 \qquad A = \frac{-1}{\omega^2 + i\omega}$$

solution is $\hat{u} = c e^{-\omega^2 y} + \frac{-1}{\omega^2 + i\omega} e^{+i\omega y}$

$$\hat{u}(0,y) = 0 \Rightarrow c = +\frac{1}{\omega^2 + i\omega}$$

$$\hat{u} = \frac{1}{\omega^2 + i\omega}\left[-e^{i\omega y} + e^{-\omega^2 y}\right]$$

$$\boxed{u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(-e^{i\omega y} + e^{-\omega^2 y})}{\omega^2 + i\omega} e^{-i\omega x}\,d\omega}$$

3. (13 points) Longitudinal waves in a large circular ring containing a small spring insert satisfy the following problem:

$$u_{xx} - u_{yy} = 0\,,\ y \ge 0,\ 0 < x \le 1\ ,\ K\big(u(0,y) - u(1,y)\big) = u_x(0,y) = u_x(1,y). \quad y \geqslant 0$$

K is the spring constant. The solution to this problem is of the form

$$u(x,y) = \sum_{n=0}^{\infty} X_n(x)\left[A_n \cos\omega_n y + B_n \sin\omega_n y\right] \quad y \ge 0,\ 0 < x \le 1$$

Are the eigenfunctions X_n orthogonal?

orth condition is (for the wave equation)

$$\left[X_n X_m' - X_m X_n'\right]_0^1 = 0$$

Eigenfns satisfy $X_n'(0) = X_n'(1) = K(X_n(0) - X_n(1))$

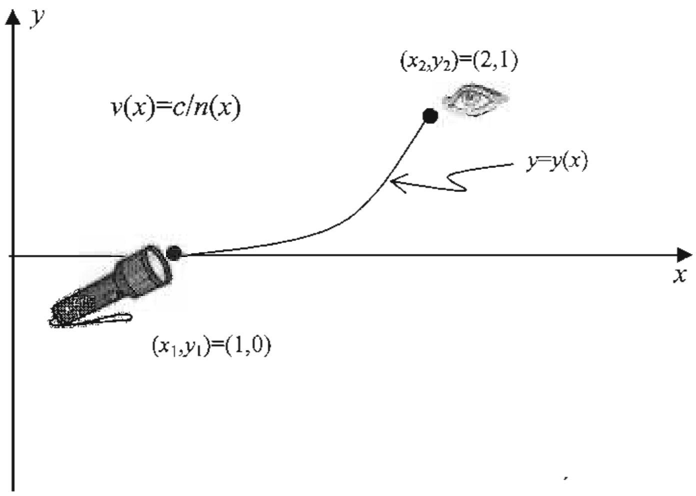
$$X_n(1) X_m'(1) - X_m(1) X_n'(1)$$
$$- X_n(0) X_m'(0) + X_m(0) X_n'(0)$$

$$= X_m'(1)\left[X_n(1) - X_n(0)\right]$$
$$- X_n'(1)\left[X_m(1) - X_m(0)\right]$$

$$= -X_m'(1)\,X_n'(1)/K + X_n'(1)\,X_m'(1) = 0$$

orthogonal!!

4. According to Fermat's principle in optics, a ray of light traveling through a medium with variable refractive $n(x)$ index follows the path $y(x)$ for which its total transit time is a minimum. Recall that the speed v of light through a given medium is the speed c of light in a vacuum divided by refractive index n: $v=c/n$.



a. Show that the total transit time for the ray of light along the path from (x_1,y_1) to (x_2,y_2) is $I[y]=\frac{1}{c}\int_{x_1}^{x_2} n(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

b. If $n(x)=1/x$ and $y(0)=0,\ y(1)=1$, find the path of light in the medium.

a) $dt = \frac{ds}{v(x)} = \frac{\sqrt{1+(y')^2}}{c/n(x)} = \frac{1}{c}\, n(x)\sqrt{1+(y')^2}$

$I[y] = \frac{1}{c}\int_{x_1}^{x_2} n(x)\sqrt{1+(y'(x))^2}\, dx$

Euler Lagrange : $(F_u = 0)$ $F_{u'}$ = constant.

$\frac{n(x)\, y'}{\sqrt{1+(y')^2}} = c = \frac{y'}{x\sqrt{1+y'^2}}$ $n = 1/x$

$(y')^2 = c^2x^2(1+(y')^2) \Rightarrow y' = \frac{cx}{\sqrt{1-c^2x^2}}$

$y(x) = \int \frac{cs}{\sqrt{1-c^2s^2}}\, ds = \frac{-1}{c}\sqrt{1-c^2x^2} + c_1$

$y(0)=0$ $-\frac{1}{c}+c_1=0$ $c,\ y(1)=1,\ \frac{1}{c}\left[-\sqrt{1-c^2}-1\right]$ ...

$1-c^2 = 1-2c+c^2$ $2c(1-c)=0$ $c=+1$ $\boxed{y = 1-\sqrt{1-x^2}}$

18