Final Exam Solutions

1. (25 points) The goal of this problem is to find a similarity solution of the form $u(x,y) = y^{\alpha} f(\eta)$, $\eta = x/y^{\beta}$ to the Eikonal equation:

$$(u_x)^2 + (u_y)^2 = 1, \ (-\infty < x < \infty, \ -\infty < y < \infty), \ u(x,x) = x.$$

a. Use the PDE to determine α, β, and an ODE satisfied by the function $f(\eta)$.
b. Use the condition $u(x,x) = x$ to determine a condition on $f(\eta)$.
c. Find solution(s) to this problem. Suggestion: try $f(\eta)=a\eta^{\gamma}$ with a and γ constants to be determined.

a) $u_x = y^{\alpha-\beta} f' \quad u_y = \alpha y^{\alpha-1} f - \beta y^{\alpha-1} \eta f'$

$$y^{2\alpha-2\beta}(f')^2 + y^{2\alpha-2}(\alpha f - \beta \eta f')^2 = 1$$

$2\alpha - 2\beta = 2\alpha - 2 = 0 \quad \alpha = 1 = \beta \quad \eta = x/y \quad u = y f(x/y)$

$$f'^2 + (f - \eta f')^2 = 1$$

b) $u(x,x) = x f(1) = x \Rightarrow f(1) = 1$

c) $f = a\eta^{\gamma} \Rightarrow f' = a\gamma\eta^{\gamma-1} \quad f(1) = 1 \Rightarrow a = 1$

Eqn is $\gamma^2 \eta^{2(\gamma-1)} + (1-\gamma)^2 \eta^{2\gamma} = 1$

$\Rightarrow \gamma = 1$ or $\gamma = 0$

Solutions are $f(\eta) = \eta$ or $f(\eta) = 1$

$u(x,y) = y\, x/y = x$ or $u(x,y) = y$

2. (25 points) The heat equation with a dissipative term is given below; μ is a nonnegative constant.

$$u_{xx} - u_y - \mu u = 0 \quad (-\infty < x < \infty,\ y \geq 0)$$

Using the Fourier transform, find the solution to the initial value problem with $u(0, y) = u_0(x)$. Express the answer as a single integral.

transform the PDE $\quad \int_{-\infty}^{\infty} e^{-i\omega x} u(x,y)\,dx = \hat{u}(\omega, y)$

$$\underbrace{\int e^{-i\omega x} u_{xx}\,dx}_{-\omega^2 \hat{u}} - \underbrace{\int e^{-i\omega x} u_y\,dx}_{\hat{u}_y} - \mu \underbrace{\int e^{-i\omega x} u\,dx}_{\hat{u}} = 0$$

(class result) $\Rightarrow \hat{u}_y + (\mu + \omega^2)\hat{u} = 0$

$$\underline{\hat{u} = e^{-(\omega^2+\mu)y}\,\hat{u}_0(\omega)} \qquad \hat{u}_0(\omega) = \hat{u}(\omega, 0) = \int_{-\infty}^{\infty} e^{-i\omega x} u(x,0)\,dx$$

$$= \int_{-\infty}^{\infty} e^{-i\omega z} u_0(z)\,dz \equiv \hat{u}_0(\omega)$$

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+i\omega x}\,\hat{u}(\omega, y)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega z} u_0(z)\,dz\; e^{-(\omega^2+\mu)y + i\omega x}\,d\omega$$

$$= \frac{1}{2\pi} e^{-\mu y}\int_{-\infty}^{\infty} u_0(z) \underbrace{\int_{-\infty}^{\infty} e^{-\omega^2 y - i\omega(z-x)}\,d\omega}_{2\pi\left(\frac{1}{2\sqrt{\pi y}} e^{-\frac{(x-z)^2}{4y}}\right)\ \text{(class)}}\,dz$$

$$\Rightarrow \boxed{u(x,y) = \frac{e^{-\mu y}}{2\sqrt{\pi y}}\int_{-\infty}^{\infty} u_0(z)\, e^{-\frac{(x-z)^2}{4y}}\,dz}$$

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~~3~~. (25 points) Consider a functional of the form $I[u] = \int_{x_1}^{x_2} F(x,u,u')dx + s_* u(x_2)^2$, defined for functions $u(x)$ with continuous second derivatives on $[x_1,x_2]$ and $u(x_1)=u_1$. Both u_1 and s_* are given constants.

a. Show that the function $u(x)$ that minimizes I satisfies the usual Euler-Lagrange equation, and derive a boundary condition at $x=x_2$ that the minimizing function must also satisfy.

v is a variation: smooth fn with $v(x_1)=0$, $v(x_2)$ unspecified

$$\frac{d}{d\alpha} I[u+\alpha v]\Big|_{\alpha=0} = 0 \quad \text{for all variations, if } u \text{ is a minimizer}$$

$$= \int_{x_1}^{x_2} (F_u v + F_{u'} v')dx + S_* \frac{d}{d\alpha}\left(u(x_2) + \alpha v(x_2)\right)^2 \Big|_{\alpha=0}$$

$$= \int_{x_1}^{x_2} (F_u v + F_{u'} v')dx + S_* 2u(x_2)v(x_2)$$

$$= F_{u'} v \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left(F_u - \frac{d}{dx}(F_{u'})\right) v\, dx + S_* 2uv\Big|_{x_2}$$

$$= \left(F_{u'} + 2S_* u\right)\Big|_{x=x_2} v(x_2) + \int_{x_1}^{x_2} \left(F_u - \frac{d}{dx}(F_{u'})\right) v\, dx = 0$$

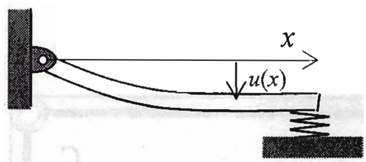
for all variations v. $\Rightarrow$ E-L hold

$$\underline{F_{u'} - \frac{d}{dx}(F_{u'}) = 0}$$

& Natural BC @ $x=x_2$ is

$$\underline{F_{u'}(x_2, u(x_2), u'(x_2)) + 2S_* u(x_2) = 0}$$

b. The functional $I[u]=\int_0^1\left[\frac{1}{2}u'(x)^2-wu(x)\right]dx+\frac{1}{2}k\left[u(1)^2\right]$ gives an approximate expression for the normalized potential energy of a thick rod distorting under its own weight. The bar is attached to a fixed point at x=0 (so that u(0)=0) and a spring at x=1. The parameters k and w are given constants.



Find the function u that minimizes the potential energy of the bar. Suggestion: do not use the first integral of the Euler-Lagrange equation. Instead, use $\frac{d}{dx}\left(F_{u'}\right)-F_u=0$. If you were not able to complete part a, solve the Euler-Lagrange equation and determine as many unknown constants as you can.

$F=\frac{1}{2}(u')^2-wu \qquad F_{u'}=u' \quad F_u=-w$

$\frac{d}{dx}(F_{u'})-F_u=0 \Rightarrow u''+w=0$

$u'=-wx+c \qquad u=-\frac{1}{2}wx^2+cx \quad (u(0)=0)$

End condition @ $x=1$ $\quad F_{u'}+\frac{1}{2}ku=0$

or $u'+ku=0$ @ $x=1$

$-w+c-\frac{1}{2}kw+kc=0$

$c(1+k)=w\left(1+\frac{k}{2}\right)$

$c=w\frac{(1+k/2)}{1+k}=\frac{w}{2}\frac{(2+k)}{1+k}$

$$\boxed{u=\frac{w}{2}\left(-\frac{1}{2}x^2+\frac{2+k}{1+k}x\right)}$$